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AN INTEGRAL EQUATION RELATING THE GENERAL TIME-DEPENDENT LIFT AND DOWNWASH DISTRIBUTIONS ON FINITE WINGS IN SUBSONIC FLOW

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SUMMARY

An integral equation for obtaining the unsteady air forces on finite wings in subsonic compressible flow is presented. This equation is applicable for any arbitrary time-dependent motion and can be utilized for flexible as well as rigid wings. The approach involves the derivation of an integral equation relating the unknown pressure distribution to any arbitrary time-dependent downwash distribution. The form of the equation is such that it should lend itself readily to modern high-speed computers for obtaining pressure distributions. Special cases of the integral equation are treated for two-dimensional incompressible flow and are presented in an appendix.

INTRODUCTION

With regard to the analytical determination of the unsteady air forces on wings undergoing sinking or pitching motion in subsonic compressible flow, most of the past efforts have been directed toward the determination of the forces on a two-dimensional wing in incompressible flow. These efforts have led to the Wagner function (ref. 1) and the Küssner function (ref. 2) which determine the unsteady forces on a wing in sinking motion and penetrating a sharp-edge gust, respectively. In references 3 and 4 approximate methods were used to obtain the forces on a few finite wings in incompressible flow. As far as the author is aware, the only work done on subsonic compressible flow, is for a two-dimensional wing in sinking and pitching motion, the results of which are presented in references 5 to 7.

The purpose of this paper is to present an integral equation for obtaining the unsteady air forces on finite wings in subsonic compressible flow. The equation is applicable for any arbitrary time-dependent motion and can be utilized for flexible as well as rigid wings. The approach involves the derivation of an integral equation relating the unknown pressure distribution to a prescribed time-dependent downwash distribution. The availability of an equation in a form which can be rapidly evaluated makes possible the use of numerical procedures to obtain the unsteady air forces which would be useful in calculating the dynamic response of airplanes to such forcing functions as those associated with gusts and blasts.

SYMBOLS

a	speed of sound
c	local wing chord
C_L	lift coefficient
$g(\ )$	time history of loading distribution (see eq. (A3))
$h(\ )$	loading distribution associated with apparent mass (see eq. (A3))
$H_n^{(2)}(\ )$	Hankel functions of second kind of order n
$J_n(\ )$	Bessel functions of first kind of order n
$K_n(\ )$	modified Bessel functions of second kind of order n
$K(x_0,y_0)$	kernel function of integral equation
k	reduced-frequency parameter, $(\omega c/2V)$
M	Mach number
p	Laplace transform variable
Δp	local lifting pressure, positive upward
q	dynamic pressure, $\rho V^2/2$

$R = \sqrt{\lambda^2 + \beta^2\left(y_0^2 + z^2\right)}$

$R_0 = \sqrt{x_0^2 + \beta^2\left(y_0^2 + z^2\right)}$

s,s'	half-chord lengths of travel $\frac{2Vt}{c}$ and $\frac{2V\tau}{c}$, respectively

$s_0 = s - s'$

S	wing area
S_0	region common to wing area S and circle $\left(Vt - x_0\right)^2 + y_0^2 = \left(\frac{Vt}{M}\right)^2$

t,τ time variables

$t_1 = \frac{M}{\beta^2 V}\left(\sqrt{x_0^2 + \beta^2 y_0^2} - Mx_0\right)$

$U(\ )$ unit step function

V free-stream velocity

w_0,w_1,w downwash functions

$x,y,z,\xi,\eta,x',\xi',\lambda$ Cartesian coordinates

$x_0 = x - \xi$

$x_0' = x' - \xi'$

$y_0 = y - \eta$

α angle of attack

$\beta = \sqrt{1 - M^2}$

ξ_L,ξ_T leading-edge and trailing-edge coordinates, respectively

$\delta(\ )$ Dirac delta function

ρ fluid density

ϕ velcoity potential

ψ acceleration potential

ω circular frequency

A bar over a quantity indicates the Laplace transform of that quantity; a bar on the integral sign $\displaystyle{-\!\!\!\!\!\!\int}$ indicates that the finite part of the integral is to be retained.

ANALYSIS

Derivation of the Integral Equation Relating the Pressure Distribution and an Arbitrary Time-Dependent Downwash

The main purpose of this analysis is to derive an integral equation that relates the unsteady pressure distribution to a known or prescribed general downwash distribution on rigid or flexible finite wings in a compressible subsonic flow. The integral equation referred to can be obtained by employing the acceleration potential to treat, by means of doublet distributions, the linearized boundary-value problems for time-dependent motion of finite wings.

The linearized partial differential equation for the acceleration potential ψ is (referred to a moving coordinate system x,y,z)

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2} - \frac{1}{a^2}\left(V\frac{\partial}{\partial x} + \frac{\partial}{\partial t}\right)^2\psi = 0 \tag{1}$$

The dependent variable in equation (1) is directly proportional to a perturbation pressure field and is related to a velocity potential ϕ as follows:

$$\psi = \frac{\partial\phi}{\partial t} + V\frac{\partial\phi}{\partial x} = -\frac{\Delta p}{\rho} \tag{2}$$

The solution of equation (2) can be obtained with the use of a Laplace transformation with respect to t. For an assumed condition of no disturbance before $t = 0$, that is, $\left(\frac{\partial\phi}{\partial t}\right)_{t=0} = 0$ it follows that

$$p\overline{\phi} + V\frac{\partial\overline{\phi}}{\partial x} = \overline{\psi} \tag{3}$$

where p is the transform variable and the bar over a quantity represents the transformed quantity - for example,

$$\overline{\psi} = \int_0^\infty e^{-pt}\,\psi(t)\,dt$$

Equation (3) can be integrated with respect to x to give

$$\overline{\phi} = \frac{1}{V}e^{-\frac{px_0}{V}}\int_{-\infty}^{x_0} e^{\frac{p\lambda}{V}}\overline{\psi}(\lambda,y_0,z;p)\,d\lambda \tag{4}$$

where $x_0 = x - \xi$ and $y_0 = y - \eta$. The lower limit of integration is chosen for convenience to satisfy the condition that ϕ vanishes as $x \to -\infty$.

A fundamental solution to equation (1) for subsonic compressible flow is (see, for example, ref. 8)

$$\psi\left(x_0,y_0,z;t\right) = \frac{1}{4\pi}\frac{\partial}{\partial z}\left[\frac{1}{R_0}f\left(t+\frac{M^2x_0}{V\beta^2}-\frac{MR_0}{V\beta^2}\right)\right] \tag{5}$$

where $R_0 = \sqrt{{x_0}^2 + \beta^2{y_0}^2 + \beta^2z^2}$ and f is an arbitrary function and represents the magnitude of a pressure doublet.

In most analyses it is convenient to use the response to a unit step function or unit impulse function to obtain the response to an arbitrary forcing function. Therefore, for convenience $f(t)$ is chosen as a unit step pressure doublet $U(t)$, where

$$\left.\begin{aligned} f(t) &= U(t) = 0 \qquad (t \leqq 0) \\ f(t) &= U(t) = 1 \qquad (t > 0) \end{aligned}\right\} \tag{6}$$

Substitution of equation (6) into equation (5) yields

$$\psi\left(x_0,y_0,z;t\right) = \frac{1}{4\pi}\frac{\partial}{\partial z}\left[\frac{1}{R_0}U\left(t+\frac{M^2x_0}{V\beta^2}-\frac{MR_0}{V\beta^2}\right)\right] \tag{7}$$

where ψ now represents the potential at $x,y,z;t$ due to a unit step pressure doublet that occurred at $t = 0$ and was located in the xy-plane at $\xi,\eta,0$. Taking the Laplace transform of equation (7) to obtain $\overline{\psi}$ and then substituting $\overline{\psi}$ into equation (4) gives the following Laplace transform of the velocity potential corresponding to the unit step function:

$$\begin{aligned} \overline{\phi} &= \frac{1}{4\pi V}e^{-\frac{px_0}{V}}\int_{-\infty}^{x_0}\frac{\partial}{\partial z}\left[\frac{1}{Rp}e^{p\left(\frac{M^2\lambda}{V\beta^2}+\frac{\lambda}{V}-\frac{MR}{V\beta^2}\right)}\right]d\lambda \\ &= \frac{1}{4\pi V}\int_{-\infty}^{x_0}\frac{\partial}{\partial z}\left[\frac{1}{Rp}e^{-\frac{p}{V}\left(x_0-\frac{\lambda}{\beta^2}+\frac{MR}{\beta^2}\right)}\right]d\lambda \end{aligned} \tag{8}$$

where $R = \sqrt{\lambda^2 + \beta^2\left({y_0}^2 + z^2\right)}$. The inverse transform of equation (8) yields

$$\phi(x,y,z;t) = \frac{1}{4\pi}\int_{-\infty}^{x_0}\frac{\partial}{\partial z}\left\{\frac{1}{R}U\left[t-\frac{1}{V}\left(x_0-\frac{\lambda}{\beta^2}+\frac{MR}{\beta^2}\right)\right]\right\}d\lambda \tag{9a}$$

which is in agreement with equation (16) of reference 9.

It should be noted that the argument of the unit step function in equation (9a) has two zeros: namely,

$$\lambda = \lambda_1 = -(Vt - x_0) + M\sqrt{(Vt - x_0)^2 + y_0^2 + z^2}$$

and

$$\lambda = \lambda_2 = -(Vt - x_0) - M\sqrt{(Vt - x_0)^2 + y_0^2 + z^2}$$

However, it can be shown that the argument is positive only when λ is greater than λ_1. Consequently equation (9a) can be rewritten as

$$\phi(x,y,z;t) = \frac{1}{4\pi} \int_{\lambda_1}^{x_0} \frac{\partial}{\partial z}\left[\frac{U(\lambda-\lambda_1)}{R}\right] d\lambda \tag{9b}$$

Substituting λ_1 into equation (9b) and performing the indicated operations yields

$$\phi(x,y,z;t) = \frac{-z}{4\pi V\left(y_0^2 + z^2\right)}\left[\frac{Vt - x_0}{\sqrt{\left(Vt - x_0\right)^2 + y_0^2 + z^2}} + \frac{x_0}{\sqrt{x_0^2 + \beta^2\left(y_0^2 + z^2\right)}}\right] U\left(x_0-\lambda_1\right) \tag{10}$$

For linear theory, the downwash in the $z = 0$ plane associated with the velocity potential ϕ can be written as

$$w_0\left(x_0,y_0;t\right) = \left(\frac{\partial\phi}{\partial z}\right)_{z=0} \tag{11}$$

which when applied to equation (10) becomes

$$w_0(x_0,y_0;t) = -\frac{1}{4\pi V}\left[\frac{Vt - x_0}{y_0^2\sqrt{(Vt - x_0)^2 + y_0^2}} + \frac{x_0}{y_0^2\sqrt{x_0^2 + \beta^2 y_0^2}}\right] U\left[Vt-M\sqrt{(Vt-x_0)^2+y_0^2}\right] \tag{12}$$

Equation (12) now represents the indicial downwash at $x,y;t$ due to a unit step pressure doublet at $\xi,\eta;0$. The downwash at $x,y;t$ due to a time-varying pressure doublet of magnitude $\frac{\Delta p(\xi,\eta;t)}{\rho}$ can be written by means of the superposition integral as

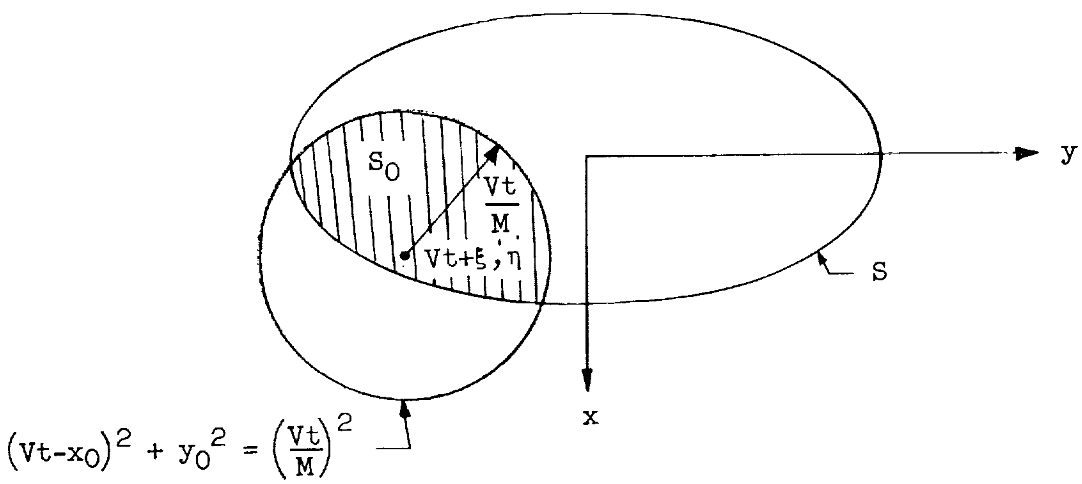
$$w_1\left(x_0,y_0;t\right) = \frac{\Delta p(\xi,\eta;t-t_1)}{\rho} w_0\left(x_0,y_0;t_1\right) + \int_{t_1}^{t} \frac{\Delta p(\xi,\eta;t-\tau)}{\rho} \frac{\partial w_0\left(x_0,y_0;\tau\right)}{\partial\tau}\, d\tau \tag{13a}$$

$$w_1(x_0,y_0;t) = \frac{\Delta p(\xi,\eta;t-t_1)}{\rho} w_0(x_0,y_0;t_1) + \int_0^{t-t_1} \frac{\Delta p(\xi,\eta;\tau)}{\rho} \frac{\partial w_0(x_0,y_0;t-\tau)}{\partial \tau} d\tau \tag{13b}$$

where $t_1 = \left(M\sqrt{x_0^2 + \beta^2 y_0^2} - M^2 x_0\right)\Big/\beta^2 V$ is the value of t for which the argument of the unit step function in equation (12) vanishes. From equation (12) it can be seen that the downwash is zero everywhere in the $z = 0$ plane except within the circle $(Vt - x_0)^2 + y_0^2 = \left(\frac{Vt}{M}\right)^2$ (the region for which the argument of the unit step function is greater than zero). The interior region of the circle is representative of a region of disturbance which is due to a doublet moving downstream with velocity V and whose waves are propagating outward at a rate equal to the speed of sound a. Therefore, by distributing pressure doublets over this area, the downwash at x,y can be obtained for any arbitrary time-dependent pressure distribution by means of the following equation:

$$w(x,y;t) = \iint_{S_0} w_1(x_0,y_0;t)\, d\xi\, d\eta \tag{14}$$

where the bars on the integral signs indicate that the finite part of the integral is to be retained and S_0 is the area which is common to both the circle $(Vt - x_0)^2 + y_0^2 = \left(\frac{Vt}{M}\right)^2$ and the area of the wing S, as can be seen in the following sketch:



Sketch 1

Substituting w_0 from equation (12) into equation (13a) and then substituting the resulting expression into equation (14) gives the following expression for the downwash:

$$w(x,y;t) = -\frac{V}{8\pi} \iint_{S_0} \left\{ \frac{\Delta p\left(\xi,\eta;t-t_1\right)}{q y_0^2} \left[\frac{Vt_1 - x_0}{\sqrt{\left(Vt_1 - x_0\right)^2 + y_0^2}} + \frac{x_0}{\sqrt{x_0^2 + \beta^2 y_0^2}} \right] \right.$$

$$\left. + V \int_{t_1}^{t} \frac{\Delta p(\xi,\eta;t-\tau)/q}{\left[\left(V\tau - x_0\right)^2 + y_0^2\right]^{3/2}} d\tau \right\} d\xi \; d\eta \qquad (15)$$

where as mentioned previously $x_0 = x - \xi$, $y_0 = y - \eta$, $t_1 = \dfrac{\left(M\sqrt{x_0^2 + \beta^2 y_0^2} - M^2 x_0\right)}{V\beta^2}$,
and S_0 is the area which is common to both the circle $\left(Vt - x_0\right)^2 + y_0^2 = \left(\frac{Vt}{M}\right)^2$ and the area of the wing S. Since equation (15) indicates that the finite part of the integral must be retained, an appropriate limiting procedure similar to the one presented in reference 10 must be devised for the regions near the singularities in order that the equation may be adapted to modern high-speed computers.

Application of Downwash Equation (15) to Special Cases

In this section it is shown that equation (15) reduces to the kernel function for the oscillating wing. In addition, it is shown that the correct value of $\Delta p/q$ is obtained for $t = 0$. An equation for three-dimensional incompressible flow and for two-dimensional compressible flow is also given.

Reduction of downwash equation (15) for oscillating wings.- For oscillating finite wings in subsonic flow the pressure coefficient can be expressed as

$$\frac{\Delta p(\xi,\eta;t)}{q} = e^{i\omega t} \frac{\Delta p(\xi,\eta)}{q}$$

where ω is the circular frequency of oscillation. By using this pressure coefficient, equation (15) can now be rewritten as

$$w(x,y;t) = -\frac{Ve^{i\omega t}}{8\pi} \iint_{S_0} \frac{\Delta p(\xi,\eta)}{q} K_0\left(x_0,y_0\right) d\xi \; d\eta$$

where

$$K_0\left(x_0,y_0\right) = \frac{1}{y_0^2} e^{-i\omega t_1} \left[\frac{Vt_1 - x_0}{\sqrt{\left(Vt_1 - x_0\right)^2 + y_0^2}} + \frac{x_0}{\sqrt{x_0^2 + \beta^2 y_0^2}} \right] + V \int_{t_1}^{t} \frac{e^{-i\omega\tau} d\tau}{\left[\left(V\tau - x_0\right)^2 + y_0^2\right]^{3/2}} \qquad (16)$$

and is the kernel function of the integral equation relating the lift and downwash distributions of oscillating finite wings in subsonic flow. In order to isolate the transient and steady-state parts of equation (16), the integral is separated into two intervals: one interval from t_1 to ∞ minus one interval from t to ∞. Then, for the steady part only

$$K(x_0,y_0) = \frac{1}{y_0^2}e^{-i\omega t_1}\left[\frac{Vt_1 - x_0}{\sqrt{(Vt_1 - x_0)^2 + y_0^2}} + \frac{x_0}{\sqrt{x_0^2 + \beta^2 y_0^2}}\right]$$

$$+ V\int_{t_1}^{\infty} \frac{e^{-i\omega\tau}d\tau}{\left[(V\tau - x_0)^2 + y_0^2\right]^{3/2}} \tag{17}$$

By means of the transformation

$$V\tau - x_0 = \frac{-\xi + M\sqrt{\xi^2 + \beta^2 y_0^2}}{\beta^2 y_0}$$

it can be shown that equation (17) is equivalent to equation (B8) of reference 11.

Evaluation of pressure coefficient as $t \rightarrow 0$.- As $t \rightarrow 0$, the radius of the circle associated with equation (15) approaches zero. (See sketch 1.) Consequently, since the area of integration approaches zero, it is permissible to assume that the pressure is uniform over this region - that is $\frac{\Delta p(\xi,\eta;t)}{q} = \frac{\Delta p(x,y;0)}{q}$. Equation (15) then takes the following form:

$$w(x,y;t) = \frac{-V}{8\pi}\frac{\Delta p(x,y;0)}{q}\int_{x-Vt\frac{1+M}{M}}^{x+Vt\frac{1-M}{M}} d\xi \int_{y-\sqrt{\left(\frac{Vt}{M}\right)^2-(Vt-x_0)^2}}^{y+\sqrt{\left(\frac{Vt}{M}\right)^2-(Vt-x_0)^2}} \left\{\frac{1}{y_0^2}\left[\frac{Vt_1 - x_0}{\sqrt{(Vt_1 - x_0)^2 + y_0^2}} + \frac{x_0}{\sqrt{x_0^2 + \beta^2 y_0^2}}\right]\right.$$

$$\left. + V\int_{t_1}^{t} \frac{d\tau}{\left[(V\tau - x_0)^2 + y_0^2\right]^{3/2}}\right\} d\eta \tag{18}$$

Evaluation of equation (18) at the limit $t \rightarrow 0$, gives

$$w(x,y;0) = \frac{MV}{4} \frac{\Delta p(x,y;0)}{q}$$

hence the well-known result

$$\frac{\Delta p(x,y;0)}{q} = \frac{4}{M} \frac{w(x,y;0)}{V} \tag{19}$$

General unsteady three-dimensional incompressible flow (M = 0).- For three-dimensional incompressible flow (M = 0) the radius of the circle (sketch 1) expands with infinite velocity so that the area of integration becomes the area of the wing. With $t_1 = 0$ equation (15) then becomes

$$w(x,y;t) = \frac{-V^2}{8\pi} \iint_S \int_0^t \frac{\frac{\Delta p(\xi,\eta;t-\tau)}{q} d\tau\, d\xi\, d\eta}{\left[\left(V\tau - x_0\right)^2 + y_0^2\right]^{3/2}} \tag{20}$$

or for two-dimensional flow

$$w(x;t) = \frac{-V^2}{4\pi} \int_{\xi_L}^{\xi_T} d\xi \int_0^t \frac{\frac{\Delta p(\xi;t-\tau)}{q}}{\left(V\tau - x_0\right)^2} d\tau \tag{21}$$

General unsteady two-dimensional subsonic flow.- For two-dimensional subsonic flow the pressure coefficient is independent of η and the limits of integration in the η-direction are dictated by the circle only. However, examination of equation (15) shows that $\Delta p/q$ is a function of ξ, η, and t_1 and, since t_1 is a function of η, equation (15) cannot be utilized to obtain two-dimensional results. Therefore, equation (12) is developed to yield two-dimensional results. By integrating equation (12) with respect to η there is obtained

$$w_0(x;t) = -\frac{1}{4\pi V} \int_{y-\sqrt{\left(\frac{Vt}{M}\right)^2-(Vt-x_0)^2}}^{y+\sqrt{\left(\frac{Vt}{M}\right)^2-(Vt-x_0)^2}} \left[\frac{Vt - x_0}{y_0^2\sqrt{\left(Vt - x_0\right)^2 + y_0^2}} + \frac{x_0}{y_0^2\sqrt{x_0^2 + \beta^2 y_0^2}}\right] d\eta$$

$$= \frac{\sqrt{(Vt)^2 - M^2\left(Vt - x_0\right)^2}}{2\pi V x_0\left(Vt - x_0\right)} \qquad \left(-\frac{Vt}{M}(1 - M) < x_0 < \frac{Vt}{M}(1 + M)\right) \tag{22}$$

By utilizing the superposition integral and distributing the doublets between $\xi = x - \frac{Vt}{M}(1 + M)$ and $\xi = x + \frac{Vt}{M}(1 - M)$, this expression for the downwash can be written as

$$w(x;t) = -\frac{V^2}{4\pi} \int_{\xi_1}^{x} d\xi \int_{\frac{Mx_0}{V(1+M)}}^{t} \frac{V\tau \frac{\Delta p(\xi;t-\tau)}{q} \, d\tau}{(V\tau - x_0)^2 \sqrt{(V\tau)^2 - M^2(V\tau - x_0)^2}}$$

$$-\frac{V^2}{4\pi} \int_{x}^{\xi_2} d\xi \int_{-\frac{Mx_0}{V(1-M)}}^{t} \frac{V\tau \frac{\Delta p(\xi;t-\tau)}{q} \, d\tau}{(V\tau - x_0)^2 \sqrt{(V\tau)^2 - M^2(V\tau - x_0)^2}} \tag{23}$$

where

$$\xi_1 = \begin{cases} x - \frac{Vt(1 + M)}{M} & \left(x - \frac{Vt(1 + M)}{M} > \xi_L\right) \\ \xi_L & \left(x - \frac{Vt(1 + M)}{M} < \xi_L\right) \end{cases}$$

and

$$\xi_2 = \begin{cases} x + \frac{Vt(1 - M)}{M} & \left(x + \frac{Vt(1 - M)}{M} < \xi_T\right) \\ \xi_T & \left(x + \frac{Vt(1 - M)}{M} > \xi_T\right) \end{cases}$$

It might be noted that for $M = 0$ equation (23) reduces to

$$w(x;t) = -\frac{V^2}{4\pi} \int_{\xi_L}^{\xi_T} d\xi \int_0^t \frac{\frac{\Delta p(\xi;t-\tau)}{q}}{(V\tau - x_0)^2} \, d\tau \tag{24}$$

which is in agreement with equation (21). A more thorough investigation of equation (24) is made in the appendix wherein it is shown that equation (24) can be expressed in a form that will yield the well-known Wagner and Küssner functions.

CONCLUDING REMARKS

The main purpose of this paper was to present an integral equation relating the downwash to a general unsteady pressure distribution. The integral expression is of a form that should lend itself readily to modern high-speed computing machines.

Expressions for three-dimensional incompressible flow and two-dimensional compressible flow are given and it can be seen for these two special cases that the amount of work involved in obtaining the pressure distributions is considerably reduced. In particular, for two-dimensional incompressible flow a method has been developed in the appendix for the rapid determination of the growth of lift for any arbitrary time-dependent downwash distribution, with special attention being given to a wing having a sudden change in angle of attack or penetrating a sharp-edge gust.

It might also be noted that the kernel function for oscillating finite wings is obtained as a special case of the integral expression.

Langley Research Center,
National Aeronautics and Space Administration,
Langley Station, Hampton, Va., October 3, 1962.

APPENDIX

DETERMINATION OF THE WAGNER AND KÜSSNER FUNCTIONS

The integral equation relating the downwash and pressure distributions on a two-dimensional wing in incompressible flow is given as equation (24) in the text and is restated here for convenience:

$$w(x;t) = -\frac{V^2}{4\pi} \int_{-c/2}^{c/2} d\xi \int_0^t \frac{\frac{\Delta p(\xi;t-\tau)}{q}}{(V\tau - x_0)^2} d\tau \tag{A1}$$

The airfoil is assumed to be moving in the negative x-direction and with the leading edge and trailing edge at $\xi_L = -c/2$ and $\xi_T = c/2$, respectively. By means of the transformations

$$s = \frac{Vt}{c/2} \qquad s' = \frac{V\tau}{c/2}$$

$$x' = \frac{x}{c/2} \qquad \xi' = \frac{\xi}{c/2}$$

$$x_0' = x' - \xi' \qquad s_0 = s - s'$$

equation (A1) becomes

$$\begin{aligned} w(x';s) &= -\frac{V}{4\pi} \int_{-1}^{1} d\xi' \int_0^s \frac{\frac{\Delta p(\xi';s_0)}{q}}{(s - x_0')^2} ds' \\ &= -\frac{V}{4\pi} \int_{-1}^{1} d\xi' \int_0^s \frac{\frac{\Delta p(\xi';s')}{q}}{(s_0 - x_0')^2} ds' \end{aligned} \tag{A2}$$

In reference 12 it has been shown that the chordwise distribution of indicial lift on a wing undergoing sinking motion or penetrating a sharp-edge gust never varies and is identical to the load distribution on the wing in steady flow. It has also been shown that associated with the lift for the sinking wing is an apparent mass term concentrated at $t = 0$. On this basis it is assumed that

$$\frac{\Delta p(\xi';s')}{q} = g(s')\sqrt{\frac{1 - \xi'}{1 + \xi'}} + h(\xi')\ \delta(s') \tag{A3}$$

where $g(s')$ is the time history of the loading distribution, $\sqrt{\frac{1 - \xi'}{1 + \xi'}}$ is the loading distribution for steady flow, $h(\xi')$ is the loading distribution associated with the apparent mass, and $\delta(s')$ is the Dirac delta function with the following properties:

$$\int_{-\infty}^{\infty} \delta(s')ds' = 1$$

and

$$\int_{-\infty}^{\infty} F(s-s')\ \delta(s')ds' = F(s)$$

The form of equation (A3) is exact, within the application of linearized theory, for the downwash considered herein. For any other downwash distribution, additional terms are required on the right-hand side of equation (A3). By substituting equation (A3) into equation (A2) and integrating both sides of equation (A2) over the chord of the wing, a simple integral equation for $g(s')$ can be obtained. The pressure distribution chosen in equation (A3) can be used to obtain the total lift and moments for a wing given a sudden change in angle of attack or penetrating a sharp-edge gust. Once the total lift and moment for a uniform downwash distribution is known, the total lift and moment for any downwash distribution can be obtained by means of existing reverse-flow theorems.

When equation (A2) in conjunction with equation (A3) is integrated with respect to x', the resulting form is

$$\int_{-1}^{1} w(x';s)dx' = -\frac{V}{4\pi} \int_{-1}^{1} dx' \ \int_{-1}^{1} d\xi' \ \int_{0}^{s} \frac{1}{(s_0 - x_0')^2}\left[g(s')\sqrt{\frac{1 - \xi'}{1 + \xi'}} + h(\xi')\ \delta(s')\right] ds' \qquad \text{(A4)}$$

Before proceeding to evaluate $g(s')$ in equation (A4), it is necessary to determine $h(\xi')$. By eliminating the integration with respect to x' and letting s approach zero, equation (A4) becomes

$$w(x';0) = -\frac{V}{4\pi} \int_{-1}^{1} \frac{h(\xi')}{(x' - \xi')^2}\, d\xi' \qquad \text{(A5)}$$

Then integrating by parts and inverting gives

$$\frac{dh(x)}{dx} = \frac{-4}{\pi V\sqrt{1 - x^2}} \int_{-1}^{1} \frac{\sqrt{1 - \xi^2}w(\xi;0)}{x - \xi}\, d\xi \qquad \text{(A6)}$$

or

$$h(x) = \frac{4}{\pi V} \int_{x}^{1} dx' \, \fint_{-1}^{1} \frac{\sqrt{1 - \xi^2}\, w(\xi;0)}{(x' - \xi)\sqrt{1 - (x')^2}} \, d\xi \tag{A7}$$

where the upper limit of integration with respect to x' is chosen to satisfy the Kutta condition on the trailing edge. It is of interest to note that for $\frac{w(\xi;0)}{V} = \alpha$ equation (A7) yields

$$h(x) = 4\alpha\sqrt{1 - x^2} \tag{A8}$$

Substituting equation (A8) into equation (A3) gives

$$\frac{\Delta p(\xi;0)}{q} = 4\alpha \, \delta(s)\sqrt{1 - \xi^2} \tag{A9}$$

which is in agreement with the result presented in reference 12.

Performing the integrations with respect to x' and ξ' on the first term in the right-hand side of equation (A4) results in the following equation:

$$\int_{-1}^{1} \frac{w(x';s)}{V} \, dx' + \frac{1}{4\pi} \int_{-1}^{1} dx' \, \fint_{-1}^{1} \frac{h(\xi')d\xi'}{(s - x_0')^2} = \frac{1}{4} \int_{0}^{s} g(s')\sqrt{\frac{s_0 + 2}{s_0}} \, ds' \tag{A10}$$

For a sudden change in angle of attack, $\frac{w(x';s)}{V} = \alpha$; substituting this expression into equation (A10) together with $h(x)$ from equation (A8) yields

$$\alpha\sqrt{s(s + 2)} = \frac{1}{4} \int_{0}^{s} g(s')\sqrt{\frac{s_0 + 2}{s_0}} \, ds' \tag{A11}$$

For a wing penetrating a sharp-edge gust $\frac{w(x';s)}{V} = \alpha U(s-x'-1)$; substituting this expression into equation (A10) together with $h(x) = 0$ yields

$$\alpha s = \frac{1}{4} \int_{0}^{s} g(s')\sqrt{\frac{s_0 + 2}{s_0}} \, ds' \tag{A12}$$

Taking the Laplace transform of equation (A11) gives

$$\frac{\alpha}{p}e^{p}K_1(p) = \frac{1}{4}e^{p}\left[K_0(p) + K_1(p)\right]\overline{g}(p) \tag{A13}$$

which, when solved for $\overline{g}(p)$, yields

$$\overline{g}(p) = \frac{4\alpha K_1(p)}{p\left[K_0(p) + K_1(p)\right]} \tag{A14}$$

where $K_n(p)$ is the modified Bessel function of the second kind of order n. An asymptotic expansion of $K_n(p)$ for large values of p (corresponding to small values of s) when substituted into equation (A14) gives

$$\overline{g}(p) \approx 2\alpha\left(\frac{1}{p} + \frac{1}{4p^2} - \frac{1}{8p^3} + \frac{7}{64p^4} + \ . \ . \ .\right)$$

or

$$g(s) = 2\alpha\left(1 + \frac{s}{4} - \frac{s^2}{16} + \frac{7s^3}{384} + \ . \ . \ .\right) \tag{A15}$$

The lift coefficient is

$$C_L = \frac{1}{2}\int_{-1}^{1}\frac{\Delta p(\xi';s)}{q}\,d\xi'$$

$$= \frac{1}{2}\int_{-1}^{1}\left[g(s)\sqrt{\frac{1 - \xi'}{1 + \xi'}} + 4\alpha\ \delta(s)\sqrt{1 - \xi'^2}\right]d\xi'$$

$$\approx \pi\alpha\left[1 + \frac{s}{4} - \frac{s^2}{16} + \frac{7s^2}{384} + \ . \ . \ . + \delta(s)\right] \tag{A16}$$

Equation (A16) now represents an approximate series expansion of the Wagner function. The first terms are also in agreement with the series expansion of the approximation

$$C_L = 2\pi\alpha\left(1 - \frac{2}{4 + s}\right)$$

given in reference 13. It is pointed out that, if in equation (A14) p is replaced by ik $\left(\text{where } k \text{ is a reduced-frequency parameter defined by } k = \frac{\omega c}{2V}\right.$ and ω is the circular frequency of oscillation$\left.\right)$, the following result is obtained:

$$g(k) = \frac{4}{ik}\left[\frac{H_1^{(2)}(k)}{H_1^{(2)}(k) + iH_0^{(2)}(k)}\right]$$

$$= \frac{4}{ik}C(k) \tag{A17}$$

where $C(k)$ is the Theodorsen function and, as pointed out in reference 13, is equal to ik times the Fourier transform of the Wagner function.

A similar analysis of equation (A12) yields the following results:

$$\bar{g}(p) = \frac{4e^{-p}}{p^2\left[K_0(p) + K_1(p)\right]} \tag{A18}$$

$$g(k) = \frac{4}{ik}\left\{C(k)\left[J_0(k) - iJ_1(k)\right] + iJ_1(k)\right\} e^{-ik} \tag{A19}$$

The expression within the brace of equation (A19) is the Sears function and again, as pointed out in reference 13, is equal to ik times the Fourier transform of the Küssner function.

It is of interest to note that the solution of equation (A11) for $g(s)$ will yield directly results that are proportional to the total lift, whereas in previous analyses it was necessary to determine the vorticity in the wake before the total lift could be obtained. This is due to the fact that the present method is based on the acceleration potential, whereas past methods were based on the velocity potential.

REFERENCES

1. Wagner, Herbert: Über die Entstehung des dynamischen Auftriebes von Tragflügeln. Z.a.M.M., Bd. 5, Heft 1, Feb. 1925, pp. 17-35.

2. Küssner, H. G.: Zusammenfassender Bericht über den instationären Auftrieb von Flügeln. Luftfahrtforschung, Bd. 13, Nr. 12, Dec. 20, 1936, pp. 410-424.

3. Jones, Robert T.: The Unsteady Lift of a Wing of Finite Aspect Ratio. NACA Rep. 681, 1940.

4. Drischler, Joseph A.: Approximate Indicial Lift Functions for Several Wings of Finite Span in Incompressible Flow as Obtained From Oscillatory Lift Coefficients. NACA TN 3639, 1956.

5. Lomax, Harvard, Heaslet, Max. A., Fuller, Franklyn B., and Sluder, Loma: Two- and Three-Dimensional Unsteady Lift Problems in High-Speed Flight. NACA Rep. 1077, 1952. (Supersedes NACA TN 2403 by Lomax, Heaslet, Sluder and TN 2387 by Lomax, Heaslet, Fuller; Contains material from TN 2256 by Lomax Heaslet, Fuller.)

6. Mazelsky, Bernard: Numerical Determination of Indicial Lift of a Two-Dimensional Sinking Airfoil at Subsonic Mach Numbers From Oscillatory Lift Coefficients With Calculations for Mach Number 0.7. NACA TN 2562, 1951.

7. Mazelsky, Bernard, and Drischler, Joseph A.: Numerical Determination of Indicial Lift and Moment Functions for a Two-Dimensional Sinking and Pitching Airfoil at Mach Numbers 0.5 and 0.6. NACA TN 2739, 1952.

8. Garrick, I. E.: Nonsteady Wing Characteristics. Aerodynamic Components of Aircraft at High Speeds. Vol. VII of High Speed Aerodynamics and Jet Propulsion, sec. F, A. F. Donovan and H. R. Lawrence, eds., Princeton Univ. Press, 1957, pp. 658-793.

9. Küssner, H. G.: General Airfoil Theory. NACA TM 979, 1941.

10. Watkins, Charles E., Woolston, Donald S., and Cunningham, Herbert J.: A Systematic Kernel Function Procedure for Determining Aerodynamic Forces on Oscillating or Steady Finite Wings at Subsonic Speeds. NASA TR R-48, 1959.

11. Watkins, Charles E., Runyan, Harry L., and Woolston, Donald S.: On the Kernel Function of the Integral Equation Relating the Lift and Downwash Distributions of Oscillating Finite Wings in Subsonic Flow. NACA Rep. 1234, 1955. (Supersedes NACA TN 3131.)

12. Lomax, Harvard: Indicial Aerodynamics. Aerodynamic Aspects. Pt. II of AGARD Manual on Aeroelasticity, ch. 6, W. P. Jones, ed., North Atlantic Treaty Organization (Paris).

13. Garrick, I. E.: On Some Reciprocal Relations in the Theory of Nonstationary Flows. NACA Rep. 629, 1938.

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